Goldbach Primes Associated With 2n

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Abstract

For \(2n = p + q\) where \(p\) and \(q\) are primes, the pair \((p, q)\) is called Goldbach pair (or Goldbach partition) and any constituent prime of a Goldbach pair for \(2n\) will be called the Goldbach prime associated with \(2n\). The Goldbach primes associated with \(2n\) are distributed evenly on both sides of \(n\). In this paper we show that the number of Goldbach primes associated with \(2n\) is odd if and only if \(n\) is prime. We also prove that if \(p\) is a Goldbach prime associated with \(2n\) then any prime \(q\) is Goldbach prime associated with \(2n + (q - p)\).

Keywords: prime, Goldbach pair, ceiling function.

AMS Mathematics Subject Classification: 11A41

Introduction

Goldbach conjecture [4, 7] is the oldest unsolved problem of Mathematics after settlement of Fermat’s Last Theorem in 1994 by Andrew Wiles [6]. It states that every even number \(\geq 4\) can be written as sum of two primes (or every even number greater than 4 can be written as sum of two odd primes). With every even number \(2n\) we assign a set, the set of Goldbach primes associated with \(2n\), given by

\[B(2n) = \{p \mid p\text{ and }2n-p\text{ are primes}\}\]

The Goldbach conjecture may now be stated as \(B(2n)\) is not empty for \(n \geq 2\). It is obvious that \(B(2n)\) is finite and its members can be written in ascending (descending) order. Occurrence of larger primes as least members of \(B(2n)\) is rare. For example, for \(n < 10^{10}\), \(B(2n)\) does not contain a prime larger than 2017 as its least member [3].

We have the following proposition

Proposition 1: For any \(p \in B(2n)\) there exists \(r\) such that \(p = n - r\) or \(p = n + r\).

Proof: Take \(r = \lfloor n - p \rfloor\).

Proposition 2: \(n - r \in B(2n)\) iff \(n + r \in B(2n)\)

Proof: \(n - r \in B(2n) \iff n - r\text{ and }2n - (n - r)\text{ are primes}
\[\iff n - r\text{ and }n + r\text{ are primes}
\[\iff n + r\text{ and }n - r\text{ are primes}
\[\iff n + r\text{ and }2n - (n + r)\text{ are primes}
\[\iff n + r \in B(2n)\]

A Goldbach pair associated with an even number \(2n\) is an ordered pair \((p, q)\), \(p \leq q\), both \(p\) and \(q\) primes, and \(p + q = 2n\). If \((p, q)\) is Goldbach pair associated with \(2n\) then [3, 5] call \(p + q\) as Goldbach partition of \(2n\).

Corollary: Each Goldbach pair for \(2n\) can be written as \((n - r, n + r)\) where \(0 \leq r < n\).

This proposition shows that the Goldbach primes associated with \(2n\) are evenly distributed on both sides of \(n\). For example
Goldbach Primes Associated with $2n$

$B(36) = \{5, 7, 13, 17, 19, 23, 29, 31\}$

\[\begin{array}{cccccccc}
5 & 7 & 13 & 17 & \downarrow & 19 & 23 & 29 & 31 \\
\uparrow & 18 & & & & & & & \\
\end{array}\]

And

$B(52) = \{5, 11, 23, 29, 41, 47\}$

\[\begin{array}{cccccccc}
5 & 11 & 23 & \downarrow & 29 & 41 & 47 \\
\uparrow & 26 & & & & & & & \\
\end{array}\]

The Goldbach pairs for 36 are (5, 31), (7, 29), (13, 23), (17, 19) and for 52 are (5, 47), (11, 41), (23, 29). All the primes between $n$ and $2n$ exist in $B(2n)$ for $n = 210$ and the number of Goldbach pairs is less than $(0.961) \left( \frac{n}{\log n} \right)$ for $n \geq 10^{24}$ [2].

The fact that first and second primes of a Goldbach pair for $2n$ are at equal distance from $n$ is also reflected in the table of addition modulo $2n$ on $B(2n)$ in the form of symmetry about main diagonal. These tables are given below for 36 and 52.

### Table 1

Addition modulo 36 on $B(36)$

<table>
<thead>
<tr>
<th>$\oplus_{36}$</th>
<th>5</th>
<th>7</th>
<th>13</th>
<th>17</th>
<th>19</th>
<th>23</th>
<th>29</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>12</td>
<td>18</td>
<td>22</td>
<td>24</td>
<td>28</td>
<td>34</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>14</td>
<td>20</td>
<td>24</td>
<td>26</td>
<td>30</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>18</td>
<td>20</td>
<td>26</td>
<td>30</td>
<td>32</td>
<td>0</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>17</td>
<td>22</td>
<td>24</td>
<td>30</td>
<td>34</td>
<td>0</td>
<td>4</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>19</td>
<td>24</td>
<td>26</td>
<td>32</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>23</td>
<td>28</td>
<td>30</td>
<td>0</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>29</td>
<td>34</td>
<td>0</td>
<td>6</td>
<td>10</td>
<td>12</td>
<td>16</td>
<td>22</td>
<td>24</td>
</tr>
<tr>
<td>31</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>12</td>
<td>14</td>
<td>18</td>
<td>24</td>
<td>26</td>
</tr>
</tbody>
</table>

### Table 2

Addition modulo 52 on $B(52)$

<table>
<thead>
<tr>
<th>$\oplus_{52}$</th>
<th>5</th>
<th>11</th>
<th>23</th>
<th>29</th>
<th>41</th>
<th>47</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>16</td>
<td>28</td>
<td>34</td>
<td>46</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>16</td>
<td>22</td>
<td>34</td>
<td>40</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>
Let $\lambda(2n)$ = number of Goldbach primes associated with $2n$. Unlike $\pi$ that assigns to $n$ the number of primes that are less than or equal to $n$ [1]. $\lambda$ is not necessarily increasing as is evident from the following table.

Table 3  
Number of Goldbach primes associated with $2n$

<table>
<thead>
<tr>
<th>2n</th>
<th>$\lambda(2n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>88</td>
<td>8</td>
</tr>
<tr>
<td>90</td>
<td>18</td>
</tr>
<tr>
<td>92</td>
<td>8</td>
</tr>
<tr>
<td>6568</td>
<td>140</td>
</tr>
<tr>
<td>6570</td>
<td>404</td>
</tr>
<tr>
<td>6572</td>
<td>140</td>
</tr>
<tr>
<td>9872</td>
<td>204</td>
</tr>
<tr>
<td>9987</td>
<td>197</td>
</tr>
<tr>
<td>10000</td>
<td>254</td>
</tr>
</tbody>
</table>

$B(2n)$ may be written as

$B(2n) = \{a_1, a_2, a_3, \ldots, a_{\lambda(2n)/2}, a_{\lambda(2n)/(\lambda(2n)/2+1)}, \ldots, a_{\lambda(2n)/2-2}, a_{\lambda(2n)/2-1}, a_{\lambda(2n)}\}$

where $a_1 < a_2 < a_3 < \ldots < a_{\lambda(2n)/2} < a_{\lambda(2n)/(\lambda(2n)/2+1)} < \ldots < a_{\lambda(2n)/2-2} < a_{\lambda(2n)/2-1} < a_{\lambda(2n)}$

and

$(a_1, a_{\lambda(2n)/2}), (a_2, a_{\lambda(2n)/2-1}), (a_3, a_{\lambda(2n)/2-2}), \ldots, (a_{\lambda(2n)/2-2}, a_{\lambda(2n)/2-1})$

are Goldbach pairs. Here $[\ldots]$ stands for ceiling function. We have

**Proposition 3:**  $\lambda(2n)$ is odd iff $n$ is prime.

**Proof:** Suppose $n$ is prime then $n$ and $2n - n$ are prime. Therefore $n \in B(2n)$ and $(n, n)$ is a Goldbach pair. If there are $m$ Goldbach pairs for $2n$, then $\lambda(2n) = 2(m - 1) + 1 = 2m - 1$.

Conversely if $\lambda(2n)$ is odd then

$[\lambda(2n)/2] = \lambda(2n)-([\lambda(2n)/2]-1)$

and hence the first and second components of the Goldbach pair

$(a_{\lambda(2n)/2}, a_{\lambda(2n)}-([\lambda(2n)/2]-1))$
are equal, while
\[ \delta_{\lambda(2n^2)} + \delta_{\lambda(2n^2)-\lambda(2n^2)-1} = 2n \]

Therefore
\[ \delta_{\lambda(2n^2)} = \delta_{\lambda(2n^2)-\lambda(2n^2)-1} = n \]

Hence \( n \) is prime.

**Proposition 4:** If \( p \in B(2n) \) then \( q \in B(2n + (q-p)) \) where \( p \) and \( q \) are primes.

*Proof:* Obvious because
\[
\begin{align*}
p \in B(2n) & \implies 2n - p \text{ is prime} \\
& \implies 2n + (q - p) \text{ is prime} \\
& \implies q \in B(2n + (q - p))
\end{align*}
\]

In particular if \( 3 \in B(2n) \) then \( 5 \in B(2n+2), 7 \in B(2n+4), 11 \in B(2n+8), \ldots \). For example since \( 3 \in B(5090) \) therefore \( 5 \in B(5092), 7 \in B(5094), 11 \in B(5098), 13 \in B(5100), 17 \in B(5104), 19 \in B(5106), \ldots \).

Primes have very peculiar behaviour. The number \( \lambda(2n) \) which represents the number of Goldbach primes associated with \( 2n \) behaves indifferently as well. However a nice thing about Goldbach primes associated with \( 2n \) is that they are evenly distributed on both sides of \( n \). Goldbach conjecture asks for existence of such primes and that \( \lambda(2n) \) is non-zero.

**References:**


